Plan

1) Intro on constructible functions

2) Proof Main thm case $K = \mathbb{C}$, $\rho\ \pi_1(X) \to GL(r, \mathbb{C})$ semi-simple & large

$$\left(\forall Z \subset X \qquad T^*_Z X \cdot T^*_X X \geq 0 \right)$$

(recall char$(K) = p$
$\exists \eta$ multivalued 1-form non-zero on Z
ϕ_η)

1) Constructible functions

Kashiwara, Maxim-Schürmann

Def Let X variety $/ \mathbb{C}$

$$CF(X) := \{ \alpha : X \to \mathbb{Z} \mid \alpha = \underbrace{\sum_i}_{\text{loc finite}} n_i \mathbb{1}_Z \ , \ \underset{\text{subvar}}{Z \subset X} \}$$

<u>stalk Euler characteristic</u>

$$\chi_x \quad D^b_c(X) \longrightarrow CF(X)$$
$$\mathcal{F} \longmapsto (x \mapsto \chi(\mathcal{F}_x))$$

<u>local Euler obstruction</u>

$Z \subset X \qquad Eu_Z \in CF(X)$

Properties
$$\begin{cases} Eu_Z(x) = 1 & \text{if } x \in Z_{sm} \\ Eu_Z(x) = 0 & x \notin Z \\ Eu_Z(x) & \text{depends on (analytic) top of sing of } Z \text{ at } x \end{cases}$$

- $Z = \bigcup_i Z_i$ ← irred comp $\Rightarrow Eu_Z = \sum_i Eu_{Z_i}$
- $\{ Eu_Z \}_{Z \subset X}$ form $\mathbb{Z}$-basis of $CF(X)$

Example. $Z = V(y^2 - x^3)$ $\qquad Eu_Z(0) = 1$

$Z = V(y^2 - x^2)$ $\qquad Eu_Z(0) = 2$ $\qquad$ not $\Lambda_Z := T^*_Z X$

Local index Formula (Kashiwara '73)

$$\begin{tikzcd} D^b_c(X) \arrow[r, "\chi_{stalk}"] \arrow[d, "CC"'] & CF(X) \arrow[dl, "CC"] \\ \mathcal{L}(X) & \end{tikzcd}$$

Lag var of $\mathcal{L}(X)$

$(-1)^{\dim Z} Eu_Z \mapsto \Lambda_Z$

i.e. $\alpha = \sum_i n_i (-1)^{\dim Z_i} Eu_{Z_i} \longrightarrow CC(\alpha) = \sum_i n_i \Lambda_{Z_i}$

$$\begin{tikzcd} T^\vee X & f^* T^\vee Y = T^\vee Y \times_Y X \arrow[l, "df^\vee"'] \arrow[d, "f_\pi"] \\ & T^\vee Y \end{tikzcd}$$

pushforward

$f: X \to Y$ proper

$\alpha \in CF(X)$ $\qquad CC(f_* \alpha) = f_{\pi *} (df^\vee)^* CC(\alpha) \in \mathcal{Z}(Y)$

Non-characteristic pullback

f is non-char wrt $\Lambda \in \mathcal{L}(Y)$ if $df^\vee|_{f_\pi^{-1}(\Lambda)}$ is proper $\iff f_\pi^{-1}(\Lambda) \cap \operatorname{Ker} df^\vee \subset f^* T^*_Y Y$

$\alpha \in CF(Y)$

if f non-char wrt $CC(\alpha)$

then $CC(f^* \alpha) = df^\vee_* f_\pi^* CC(\alpha)$

Def $\alpha \in CF(X)$ is CC-effective if

$CC(\alpha) = \sum_i n_i \Lambda_Z$ $\quad$ w/ $n_i \geq 0$

2) Proof of main thm

Theorem (1.5) X proj mfd

$\rho: \pi_1(X) \longrightarrow GL(r, \mathbb{C})$ large and semisimple

$\forall Z \subset X$ $\qquad \Lambda_Z \cdot T^*_X X \geq 0$

$\qquad\qquad \shortparallel$

$\qquad\qquad \deg(\Lambda_Z)$

Outline of proof

Prop A (4.72) Under these assumptions

$\exists\ \sigma: \pi_1(X) \to GL(N, \mathbb{C})$ semi-simple, underlying a $\mathbb{C}$-VHS

Recall

large $\forall Z \subset X$

$\pi_1(Z_{norm}) \to \pi_1(X) \xrightarrow{\rho} GL(N, \mathbb{C})$

inf image

and η multivalued 1-form st $\forall Z \subset X$

Either 1) $\eta_{|Z}$ non-trivial

2) the pullback ~~of~~ of σ to Z_{norm} is large & discrete monodromy

Idea Deng-Yamanoi-Katzarkov '23 Th 1.28

$S_{fac,r}$ $X \to S_{fac,r}$ sin stein fact Katzarkov-Eyssidieux red

$\forall Z \subset X$

$S_{fac,r}(Z) = \{pt\} \Rightarrow$ situation 2

or non-zero η on $Z \Rightarrow$ situation 1 (Cor 3.10)

Prop B (4.1) $Z \subset X$

ϱ $\pi_1(X) \to GL(N, \mathbb{C})$ as in part 2 of Prop A

($\mathbb{C}$-VHS, res to Z_{norm} large & disc mon)

Then $\deg(\Lambda_Z) \geq 0$

Proof of main thm We apply Prop A inductively

if 1 $\Rightarrow$ reduce $\dim Z$ using ϕ_η

if 2 $\Rightarrow$ done with Prop B

Proof of Prop B

$$\Gamma = \mathrm{Im}\left(\pi_1(Z_{norm}) \longrightarrow \pi_1(X) \xrightarrow{\sigma} GL(N, \mathbb{C}) \right)$$

Step 1. reduce Γ tors-free

by Selberg $\sigma(\pi_1(X))$ has a tors-free sub gp of finite index

$$\begin{tikzcd} \tilde{Z} \arrow[r] \arrow[d] & \tilde{X} \arrow[d, "\pi"] \\ Z \arrow[r, hook] & X \end{tikzcd}$$

etal cover associated to it

$$\deg(\Lambda_{\tilde{Z}}) = \chi(Eu_{\tilde{Z}}) = \chi(\pi_* Eu_{\tilde{Z}}) = \chi(d\, Eu_Z) = d \deg \Lambda_Z$$

global index Formula. $\chi(\alpha) = CC(\alpha)\ T^*_X X = \deg(CC(\alpha))$

Now let $\pi . \tilde{X} \to X$ induced by $\mathrm{Ker}(\sigma)$

$$\Gamma = \pi_1(Z_{norm}) / \mathrm{Ker}(\sigma) \cap \pi_1(Z_{norm})$$

$$\begin{tikzcd} \tilde{Z}_{norm} \arrow[r, "\tilde{p}"] \arrow[d, "\pi_{Z_{norm}}"] & \tilde{Z} \arrow[r, hook, "\tilde{\iota}"] \arrow[d, "\pi_Z"] & \tilde{X} \arrow[r, "\tilde{\Phi}"] \arrow[d, "\pi"] & D \arrow[d] \\ Z_{norm} \arrow[r, "p"] & Z \arrow[r, hook, "i"] & X & \Gamma \backslash D \end{tikzcd}$$

period map ass to $\mathbb{C}$-VHS ass to σ

induced by $\mathrm{Ker}(\sigma) \cap \pi_1(Z_{norm})$

$$\tilde{X} = X_{univ} / \mathrm{Ker}\,\sigma$$

$\tilde{\Phi}$ finite $\tilde{\tau}\ \tilde{p}_* \pi^*_{Z_{norm}} \delta = \tilde{\pi}_Z^* p_* \delta$

Ker(ς) ∩ π₁ Znorm

$$\begin{tikzcd}
\cdot \arrow[d, "\pi_{Z_{norm}}"] & \cdot \arrow[d, "\pi_Z"] & \cdot \arrow[d, "\pi"] & \cdot \arrow[d] \\
Z_{norm} \arrow[r, "p"] \arrow[rrr, bend right, "\phi"'] & Z \arrow[r, hook, "i"'] & X & \Gamma\backslash D
\end{tikzcd}$$

deck transf gp $= \Gamma$

$\tilde{X} = X_{univ} / \operatorname{Ker} \varsigma$

$\pi_1(\tilde{X}) = \operatorname{Ker}\varsigma$

$\tilde{\phi} \simeq \hat{p}_* \pi_{Z_{norm}}^* \delta = \cancel{\tilde{\pi}_Z^*} p_* \delta)$ — finite — CC eff

If Z normal. C Lehn $\underset{\substack{\Gamma \text{ dis} \\ \text{tors free}}}{\Longrightarrow}$ ϕ is finite & $\phi(Z)$ horizontal sub of $\Gamma\backslash D$

$T_{\phi(Z)} \subset T^h_{\Gamma\backslash D}$

$\Longrightarrow \deg(\Lambda_{\phi(Z)}) \geq 0$

$\Longrightarrow \deg(\Lambda_Z) \geq 0$

$\deg(\Lambda_{Z_{norm}}) \geq 0$

Def δ-function $\qquad p \; Z_{norm} \to Z$

$x \in Z_{norm}$

$U \ni x$ small open $\qquad \delta(x) = Eu_{p(U)}(p(x)) \; (-1)^{\dim Z}$

Prop. $p_*(\delta) = Eu_Z \cdot (-1)^{\dim Z}$

Zariski's Main Theorem. $\# p^{-1}(x) = \#$ branches of Z at x

$Z \supset U \ni x$

$U = \bigcup U_i$ ← branches at x

$p^{-1}(U) = \bigcup_i \tilde{U}_i \qquad p(\tilde{U}_i) = U_i$

disjoint union in Z_{norm}

$\Rightarrow (-1)^d p_* \delta(x) = \sum_i Eu_{p(\tilde{U}_i)}(x) = \sum_i Eu_{U_i}(x) = Eu_Z(x)$

Lem $\phi_*(\delta)$ is CC-eff

pf $p_*(\delta) = (-1)^{\dim Z} Eu_Z$ is CC-eff

$\deg(\Lambda_Z) = \chi((-1)^{\dim Z} Eu_Z) = \chi(\delta) = \chi(\phi_* \delta) = \deg(\sum_i n_i \Lambda_{Y_i})$ $\qquad n_i \geq 0$

$\geq 0 \qquad\qquad Y_i \subset \phi(Z_{norm})$

↑ horizontal